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# On a method for solving a two-dimensional nonlinear integral equation of the second kind

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ABSTRACT

In this article, the existence of at least one solution of a nonlinear integral equation of the second kind is proved. The degenerate method is used to obtain a nonlinear algebraic system, where the existence of at least one solution of this system is discussed. Finally, computational results with error estimates are obtained using Maple software.



## 1. Introduction

Integral equations of various types and kinds play an important role in many branches of linear and nonlinear functional analysis and their applications in mathematical physics, engineering, and contact problems in the theory of elasticity; see [1,2]. Therefore, many different methods have been established, and they are used to solve nonlinear integral equations (NIEs) of the first and the second kind; see [3–5]. In [6], Brunner et al. introduced a class of methods depending on some parameters to obtain numerically the solution of an Abel integral equation of the second kind. Linear multistep methods were applied by Kauthen [7] to obtain numerically the solution of a singular nonlinear Volterra integral equation (NVIE). In [8], a fast Runge–Kutta method is presented to solve a nonlinear convolution system of Volterra integral equations. Also, in [9], Kilbas and Saigo used an asymptotic method to obtain numerically the solution of a nonlinear Abel–Volterra integral equation. In [10], Orsi used a product Nyström method, as a numerical method, to obtain the solution of an NVIE when its kernel takes logarithmic and Carleman forms. Bannas and Emmanuele, in [11] and [12], respectively, studied the NIE in $L_1[-1, 1]$; their analysis depended on the technique of non-compactness. In [13], Abdou et al. proved the existence of an integrable solution of an NIE of the second kind, by using the Schauder fixed point theorem. Guoqiang et al., in [14], obtained numerically the solution of a two-dimensional NVIE by collocation and iterated collocation methods. In [15], Guoqiang and Jiong analyzed the existence of an asymptotic error expansion of the Nyström solution for a two-dimensional NIE of the second kind. In [16,17], the authors used the Toeplitz matrix method and obtained the numerical solution of an NIE in the space $L_2[-1, 1]$ and $L_p[-1, 1]$, respectively.

In this work, we use a degenerate method (DM) to obtain the solution of an NIE with a continuous kernel. In Section 2, the existence of at least one solution of an $n$-dimensional NIE is discussed and proved, using the Schauder fixed point theorem.

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In section three, we use a DM to obtain a nonlinear algebraic system (NAS). The consistency of the obtained NAS and NIE is considered. In section four, the existence of at least one solution of the NAS is discussed and proved. Finally, in section five, many examples are solved to explain the method in linear and nonlinear cases.

## 2. The existence of at least one solution of a two-dimensional NIE

Consider the equation

$$\mu\Phi(x,y)-\lambda\int_0^1\int_0^1 K(x,u;y,v)\Psi(u,v,\Phi(u,v))\mathrm{d}v\mathrm{d}u=F(x,y). \tag{1}$$

Here, $K(x,u;y,v)$ and $F(x,y)$ are known functions in the space $L_2([0,1]\times[0,1])$, and they are called the kernel and the free term of the NIE, respectively. Also, $\Psi(u,v,\Phi(u,v))$ is a known continuous function, while $\Phi(x,y)$ is unknown. The constant $\lambda$ has a physical meaning that may be complex, while the constant $\mu$ defines the kind of NIE.

Define the following integral operator:

$$W\Phi=\lambda\int_0^1\int_0^1 K(x,u;y,v)\Psi(u,v,\Phi(u,v))\mathrm{d}u\mathrm{d}v. \tag{2}$$

So, Eq. (1) can be written in operator form as

$$\mu W^*\Phi=F+W\Phi. \tag{3}$$

**Theorem 1.** *The integral equation (1) has at least one solution under the following conditions.*

(i) *The kernel $K(x,u;y,v)$ satisfies*

$$\left[\int_0^1\int_0^1\int_0^1\int_0^1 |K(x,u;y,v)|^2\mathrm{d}x\mathrm{d}u\mathrm{d}y\mathrm{d}v\right]^{\frac{1}{2}}=C,$$

*where $C$ is a small enough constant.*

(ii) *The given function $F(x,y)$ and its partial derivatives with respect to $x,y$ belong to $L_2([0,1]\times[0,1])$ space, and its norm is defined as*

$$\|F(x,y)\|_2=\left[\int_0^1\int_0^1 |F(x,y)|^2\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}}=H,$$

*where $H$ is a constant.*

(iii) *The given function $\Psi(x,y,\Phi(x,y))$, for any two functions $\Phi_1(x,y)$ and $\Phi_2(x,y)\in L_2[0,1]\times L_2[0,1]$, satisfies*

$$\left[\int_0^1\int_0^1 |\Psi(x,y,\Phi_1(x,y))-\Psi(x,y,\Phi_2(x,y))|^2\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}}\le\epsilon$$

*if*

$$\|\Phi_1(x,y)-\Phi_2(x,y)\|\le\delta(\epsilon),\quad 0<\epsilon<1.$$

*The proof of the theorem can be obtained directly from the following lemmas.*

**Lemma 1.** *Under conditions* (i)–(iii) *, the operator $W^*$ maps the ball $S_\alpha\in L_2([0,1]\times[0,1])$ into itself.*

**Proof.** In the light of Eqs. (2) and (3), we get

$$\|W^*\Phi(x,y)\|\le\frac{1}{\mu}\|F(x,y)\|+\left|\frac{\lambda}{\mu}\right|\left\|\int_0^1\int_0^1 K(x,u;y,v)\Psi(u,v,\Phi(u,v))\mathrm{d}u\mathrm{d}v\right\|.$$

Applying the Cauchy–Schwarz inequality, then using conditions (i)–(iii), we obtain

$$\|W^*\Phi(x,y)\|\le\frac{H}{\mu}+\left|\frac{\lambda}{\mu}\right|CE,\quad(\mu\ne0). \tag{4}$$

Inequality (4) shows that the operator $W^*$ maps the ball $S_\alpha$ into itself, where $\alpha=\frac{1}{\mu}[H+\lambda CE]$.

From the second term of inequality (4), we deduce that the integral operator $W\Phi(x,y)$ is bounded in the space $L_2([0,1]\times[0,1])$. Therefore, $W^*\Phi(x,y)$ is also bounded. □

**Lemma 2.** *If conditions* (i)–(iii) *are verified, then the operator $W^*$ is continuous in $S_\alpha$.*

**Proof.** Let $\Phi_1(x,y)$ and $\Phi_2(x,y)$ be any two functions in $S_\alpha$. Therefore,

$$\|W^*\Phi_1(x,y) - W^*\Phi_2(x,y)\| \le \left|\frac{\lambda}{\mu}\right| \|K(x,u;y,v)[\Psi(u,v,\Phi_1(u,v)) - \Psi(u,v,\Phi_1(u,v))]\mathrm{d}u\mathrm{d}v\|.$$

Applying the Cauchy–Schwarz inequality, then using conditions (i)–(iii), the previous inequality becomes

$$\|W^*\Phi_1(x,y) - W^*\Phi_2(x,y)\| \le \left|\frac{\lambda}{\mu}\right| C\epsilon = \epsilon^*,$$

which implies the continuity of $W^*$ in the ball $S_\alpha$. □

**Lemma 3.** *Suppose that $\{K_{n,m}(x,u;y,v)\}$ is a sequence of continuous functions such that*

$$\lim_{n,m\to\infty}\left[\int_0^1\int_0^1\int_0^1\int_0^1 \left|K_{n,m}(x,u;y,v) - K(x,u;y,v)\right|^2 \mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}} = 0. \tag{5}$$

*Then, there exist positive integers $n_0, m_0$, such that, for $n > n_0, m > m_0$, in general $n \ne m$, after neglecting the very small constants, we have*

$$\left[\int_0^1\int_0^1\int_0^1\int_0^1 |K_{n,m}(x,u;y,v)|^2 \mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}} \le C. \tag{6}$$

**Proof.**

$$\left[\int_0^1\int_0^1\int_0^1\int_0^1 |K_{n,m}(x,u;y,v)|^2 \mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}} \le \left[\int_0^1\int_0^1\int_0^1\int_0^1 \{|K_{n,m}(x,u;y,v) - K(x,u;y,v)|^2 \right.$$

$$\left. + 2|K_{n,m}(x,u;y,v) - K(x,u;y,v)||K(x,u;y,v)| + |K(x,u;y,v)|^2\}\mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}}.$$

Hence, for each $n > n_0$, $m > m_0$, using (5) and condition (i), formula (6) is verified after neglected a small constant. □

**Lemma 4.** *If conditions* (i)–(iii) *are satisfied, then the sequence of operators*

$$W^*_{n,m}\Phi(x,y) = \frac{1}{\mu}F(x,y) + \frac{\lambda}{\mu}\int_0^1\int_0^1 K_{n,m}(x,u;y,v)\Psi(u,v,\Phi(u,v))\mathrm{d}u\mathrm{d}v, \tag{7}$$

*maps the largest ball $S_\alpha$ into itself for each $n > n_0$, $m > m_0$.*

**Proof.** Formula (7) gives us

$$\|W^*_{n,m}\Phi(x,y)\| \le \frac{H}{\mu} + \left|\frac{\lambda}{\mu}\right| CE = \alpha.$$

Therefore, $W^*_{n,m}$ maps the ball $S_\alpha$ into itself.

To prove the continuity of $W^*_{n,m}$, we choose any two functions $\Phi_1(x,y)$, $\Phi_2(x,y)$ in $S_\alpha$. Then, from (7), after applying the Cauchy–Schwarz inequality and using conditions (i) and (iii), we obtain

$$\|W^*_{n,m}\Phi_1(x,y) - W^*_{n,m}\Phi_2(x,y)\| \le \left|\frac{\lambda}{\mu}\right| C\epsilon = \epsilon^* \quad \forall n > n_0,\ m > m_0. \quad \square$$

**Lemma 5.** *Under the same conditions* (i)–(iii) , *the set $W^*(S_\alpha)$ is compact.*

**Proof.** From Eqs. (2), (3) and (7), we get

$$\|W^*_{n,m}\Phi(x,y) - W^*\Phi(x,y)\| = \left|\frac{\lambda}{\mu}\right| \left\|\int_0^1\int_0^1 [K_{n,m}(x,u;y,v) - K(x,u;y,v)]\Psi(u,v,\Phi(u,v))\mathrm{d}u\mathrm{d}v\right\|.$$

Hence, using condition (iii) yields

$$\|W^*_{n,m}\Phi(x,y) - W^*\Phi(x,y)\| \le \left|\frac{\lambda}{\mu}\right| E\left[\int_0^1\int_0^1\int_0^1\int_0^1 |K_{n,m}(x,u;y,v) - k(x,u;y,v)|^2 \mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}}.$$

Also, from (5), we have the following condition:

$$\|W^*_{n,m}\Phi(x,y) - W^*\Phi(x,y)\| = 0, \quad \text{as } n, m \to \infty. \tag{8}$$

To prove the compactness of $W^*$, we let $\{\Phi_{n,m}(x,y)\}$ be any sequence in $S_\alpha$. Then, we can choose a subsequence $\{\Phi_{n_1,m}(x,y)\}$ such that $\{W^*_{n_1,m}\Phi_{n_1,m}(x,y)\}$ converges. From that subsequence, we can extract a new subsequence $\{\Phi_{n_1,m_1}(x,y)\}$ in which $\{W^*_{n_1,m_1}\Phi_{n_1,m_1}(x,y)\}$ converges, and so on. Thus, we obtain a chain of subsequences,

$$\{\Phi_{n,m}(x,y)\} \supset \{\Phi_{n_1,m}(x,y)\} \supset \{\Phi_{n_1,m_1}(x,y)\} \supset \cdots \supset \{\Phi_{n_j,m_l}(x,y)\} \supset \cdots$$

such that the sequence $\{W^*_{n_i,m_k}\Phi_{n_j,m_l}(x,y)\}$ converges for all $i = 1, 2, \ldots, j$ and $k = 1, 2, \ldots, l$. Finally, we pick the sequence $\{\Phi_{n_n,m_m}(x,y)\}$, which is a subsequence of every $\Phi_{n_i,m_k}$ except for a finite number of elements, and clearly $\{W^*_{n_i,m_k}\Phi_{n_n,m_m}\}$ converges for every $i$, $k$. Now, since

$$\|W^*_{n_i,m_k}\Phi_{n_n,m_m} - W^*_{n_i,m_k}\Phi_{p_p,q_q}\| \to 0 \quad \text{as } m, n, p, q \to \infty,$$

for large $j$, $k$, and from (8), we get

$$\|W^*\Phi_{n_n,m_m} - W^*\Phi_{p_p,q_q}\| \le 2\epsilon, \quad \forall n, p > n_0(\epsilon),\ m, q > m_0(\epsilon).$$

Hence, $\{W^*\Phi_{n,m}\}$ is a Cauchy sequence, so $W^*(S_\alpha)$ is compact.

According to the previous lemmas, we see that $W^*$ is a continuous operator that maps a closed convex set $S_\alpha$ in the space $L_2([0,1]\times[0,1])$ into itself and that $W^*(S_\alpha)$ is a compact set. So, by the Schauder fixed point theorem, $W^*$ has at least one fixed point in $S_\alpha$, and so Theorem 1 is proved. In fact, this theorem could be proved for arbitrary dimensions using the same arguments, the Cauchy–Schwarz inequality and the Minkowski inequality. □

## 3. Degenerate kernel method

Suppose that the approximate kernel $K_{n,m}(x,u;y,v)$ takes the form

$$K_{n,m}(x,u;y,v) = \sum_{i=1}^{n}\sum_{j=1}^{m} \eta_i(x)\zeta_i(u)\beta_j(y)\gamma_j(v), \tag{9}$$

where

$$|K_{n,m}(x,u;y,v) - K(x,u;y,v)| \to 0 \quad \text{as } n, m \to \infty.$$

Therefore, the integral equation (1) yields

$$\mu\Phi_{n,m}(x,y) - \lambda\int_0^1\int_0^1 K_{n,m}(x,u;y,v)\Psi(u,v,\Phi_{n,m}(u,v))\mathrm{d}v\mathrm{d}u = F(x,y) + R_{n,m}, \tag{10}$$

where $R_{n,m}$ is the error.

**Definition 1.** The two-dimensional degenerate kernel method is said to be convergent of order $r_1 + r_2$ in the domain $L_2([0,1]\times[0,1])$ if and only if, for large $n$, $m$, there exists a constant $D > 0$ independent of $n$, $m$ such that

$$\|\Phi(x,y) - \Phi_{n,m}(x,y)\| \le Dn^{-r_1}m^{-r_2}.$$

Using (9) in (10), we have

$$\mu\Phi_{n,m}(x,y) - \lambda\sum_{i,j}^{n,m}\eta_i(x)\beta_j(y)\int_0^1\int_0^1 \zeta_i(u)\gamma_j(v)\Psi(u,v,\Phi_{n,m}(u,v))\mathrm{d}u\mathrm{d}v = F(x,y).$$

Assume the unknown constants

$$A_{ij} = \int_0^1\int_0^1 \zeta_i(u)\gamma_j(v)\Psi(u,v,\Phi_{n,m}(u,v))\mathrm{d}u\mathrm{d}v, \tag{11}$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Hence, formula (10) becomes

$$\Phi_{n,m}(x,y) = \frac{\lambda}{\mu}\sum_{i,j}^{n,m}\eta_i(x)\beta_j(y)A_{ij} + \frac{1}{\mu}F(x,y), \quad (\mu \ne 0). \tag{12}$$

To determine the matrix elements $A_{ij}$, we substitute (12) into (11), to get the following NAS:

$$\bar{A} = A_{ij} = \int_0^1\int_0^1 \zeta_i(u)\gamma_j(v)\Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}^{n,m}\eta_l(u)\beta_k(v)A_{lk}\right)\right)\mathrm{d}u\mathrm{d}v, \tag{13}$$

where $\bar{A}$ is defined as

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nm} \end{pmatrix}.$$

Assume the following operator:

$$G_{ij}(\bar{A}) = \int_0^1 \int_0^1 \zeta_i(u)\gamma_j(v)\Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}^{n,m} \eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right) \mathrm{d}v\mathrm{d}u. \tag{14}$$

Formula (14) represents an NAS that can be written in vector notation as

$$\bar{A} = \bar{G}(\bar{A}),$$

where

$$\bar{G} = \begin{pmatrix} G_{11}(\bar{A}) & G_{12}(\bar{A}) & \dots & G_{1m}(\bar{A}) \\ G_{21}(\bar{A}) & G_{22}(\bar{A}) & \dots & G_{2m}(\bar{A}) \\ \vdots & \vdots & & \vdots \\ G_{n1}(\bar{A}) & G_{n2}(\bar{A}) & \dots & G_{nm}(\bar{A}), \end{pmatrix}$$

and the elements of $\bar{A}$ are given by Eq. (13).

**Theorem 2.** *Under the same assumptions of Theorem 1, the sequence $\Phi_{n,m}$ converges to the solution $\Phi(x, y)$ of Eq. (1) in the space $L_2([0, 1] \times [0, 1])$.*

**Proof.** From (1) and (10), and after using conditions (i)–(iii) and

$$\|K(x, u; y, v) - K_{n,m}(x, u; y, v)\| = 0, \quad \text{as } n, m \to \infty,$$

we get

$$\|\Phi(x, y) - \Phi_{n,m}(x, y)\| = \epsilon, \quad \text{as } n, m \to \infty. \quad \square$$

## 4. The existence of at least one solution of the NAS

**Theorem 3.** *Under the following conditions:*

$$\left[\sum_{i,j}\int_0^1\int_0^1 |\zeta_i(u)\gamma_j(v)|^2 \mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}} \cdot \left[\sum_{i,j}\int_0^1\int_0^1 |\eta_i(u)\beta_j(v)|^2 \mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}} = C^*, \tag{15}$$

*$C^*$ is a small (enough) constant.*

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^1\int_0^1 |\Psi(u, v, \varphi(u, v, A_{ij}))|^2 \mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}} \le E^*, \tag{16}$$

*$E^*$ is a constant and for two vectors $\bar{A} = (A_{ij})$, $\bar{B} = (B_{ij}) \in l_2 \times l_2$ space, we assume*

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\int_0^1\int_0^1 |\Psi(u, v, \varphi(u, v, A_{ij})) - \Psi(u, v, \varphi(u, v, B_{ij}))|^2 \mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}} \le \epsilon^*, \tag{17}$$

*where*

$$\|\bar{A} - \bar{B}\| = \left[\sum_{i=1}^{n}\sum_{j=1}^{m} |A_{ij} - B_{ij}|^2\right]^{\frac{1}{2}} \le \delta(\epsilon^*).$$

*Then the NAS ((13) or (14)) has at least one solution in the space $l_2$.*

To prove this theorem, we must consider the following lemmas.

**Lemma 6.** *Under conditions (15) and (16), the operator $\bar{G}$ of (14) maps the ball $S^*_\beta$, in the space $l_2 \times l_2$, into itself.*

**Proof.** From Eq. (14), we have

$$|G_{ij}(\bar{A})| \le \int_0^1 \int_0^1 |\zeta_i(u)\gamma_j(v)| \left| \Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k} \eta_l(u)\beta_k(v)A_{lk}\right)\right)\right| \mathrm{d}u\mathrm{d}v.$$

Summing over $i, j$, then applying the Cauchy–Minkowski inequality, we get

$$\left[\sum_{i,j}^{n,m} |G_{ij}|^2\right]^{\frac{1}{2}} \le \left[\sum_{i,j} \int_0^1 \int_0^1 |\zeta_i(u)\gamma_j(v)|^2 \mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}} \left[\sum_{i,j} \int_0^1 \int_0^1 \left| \Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k} \eta_l(u)\beta_k(v)A_{lk}\right)\right)\right|^2 \mathrm{d}u\mathrm{d}v\right]^{\frac{1}{2}}.$$

Letting $n, m \to \infty$, using (15) and (16), we have

$$\|\bar{G}(\bar{A})\| \le \left|\frac{\lambda}{\mu}\right| C^* E^*, \quad (\mu \ne 0).$$

Hence, $\bar{G}$ is a bounded operator in Banach space $l_2 \times l_2$ which maps the ball $S^*_\beta$ into itself, where $\beta = |\frac{\lambda}{\mu}| C^* E^*$. □

**Lemma 7.** *Under conditions (17), $\bar{G}$ is continuous in Banach space $l_2 \times l_2$.*

**Proof.** Let $\bar{A}$ and $\bar{B}$ be any two vectors in $S^*_\beta$. Therefore,

$$|G_{ij}(\bar{A}) - G_{ij}(\bar{B})| \le \int_0^1 \int_0^1 |\zeta_i(u)\gamma_j(v)| \left| \Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k} \eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right) - \Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k} \eta_l(u)\beta_k(v)\bar{B}_{lk}\right)\right)\right| \mathrm{d}u\mathrm{d}v.$$

Summing over $i, j$, applying the Cauchy–Minkowski inequality, letting $n, m \to \infty$, and then using (15) and (16), we obtain

$$\|\bar{G}(\bar{A}) - \bar{G}(\bar{B})\| \le \left|\frac{\lambda}{\mu}\right| C^* \epsilon^* = \epsilon,$$

where $\|\bar{A} - \bar{B}\| \le \delta(\epsilon^*)$. Therefore, $\bar{G}$ is a continuous operator. □

**Lemma 8.** *The sequence of operators $\bar{G}_{p,q}$ maps the set $S^*_\beta$ continuously into itself, where*

$$\bar{G}_{p,q} = \begin{pmatrix} (G_{11})_{p,q}(\bar{A}) & (G_{12})_{p,q}(\bar{A}) & \dots & (G_{1m})_{p,q}(\bar{A}) \\ (G_{21})_{p,q}(\bar{A}) & (G_{22})_{p,q}(\bar{A}) & \dots & (G_{2m})_{p,q}(\bar{A}) \\ \vdots & \vdots & & \vdots \\ (G_{n1})_{p,q}(\bar{A}) & (G_{n2})_{p,q}(\bar{A}) & \dots & (G_{nm})_{p,q}(\bar{A}) \end{pmatrix},$$

*and*

$$|(G_{ij})_{p,q}(\bar{A})| = \int_0^1 \int_0^1 \zeta_i(u)\gamma_j(v)\Psi_{p,q}\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k} \eta_l(u)\beta_k(v)A_{lk}\right)\right) \mathrm{d}u\mathrm{d}v. \tag{18}$$

**Proof.** In the light of Eq. (18), we have

$$|(G_{ij})_{p,q}(\bar{A})| \le \int_0^1 \int_0^1 |\zeta_i(u)\gamma_j(v)| \left| \Psi_{p,q}\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k} \eta_l(u)\beta_k(v)A_{lk}\right)\right)\right| \mathrm{d}u\mathrm{d}v.$$

Summing over $i, j$, applying the Cauchy–Minkowski inequality, letting $n, m \to \infty$, and then using (15) and (17), we get

$$\|(\bar{G})_{p,q}(\bar{A})\| \le \left|\frac{\lambda}{\mu}\right| C^* E^*.$$

Hence, $(\bar{G})_{p,q}$ is a bounded operator in the space $l_2 \times l_2$ which maps the set $S^*_\beta$ into itself, where $\beta = |\frac{\lambda}{\mu}|C^*E^*$.

For $\bar{A}$ and $\bar{B} \in S^*_\beta$, we have

$$|(G_{ij})_{p,q}(\bar{A}) - G_{ij}(\bar{B})| \leq \int_0^1 \int_0^1 |\zeta_i(u)\gamma_j(v)| \left| \Psi_{p,q}\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}\eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right) - \Psi_{p,q}\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}\eta_l(u)\beta_k(v)\bar{B}_{lk}\right)\right)\right| \mathrm{d}u\mathrm{d}v.$$

Summing over $i, j$, then applying the Cauchy–Minkowski inequality, letting $n, m \to \infty$, and using (15) and (17), we have

$$\|(\bar{G})_{p,q}(\bar{A}) - (\bar{G})_{p,q}(\bar{B})\| \leq \left|\frac{\lambda}{\mu}\right| C^*\epsilon^* = \epsilon,$$

where $\|\bar{A} - \bar{B}\| \leq \delta(\epsilon^*)$. Therefore, $(\bar{G})_{p,q}$ is a continuous operator. □

**Lemma 9.** *Under conditions (15) and (16), the set $G(S^*_\beta)$ is compact.*

**Proof.** From (18) and (14), we have

$$|(G_{ij})_{p,q}(\bar{A}) - G_{ij}(\bar{A})| = \left| \int_0^1 \int_0^1 \zeta_i(u)\gamma_j(v) \left[ \Psi_{p,q}\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}^{n,m}\eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right) - \Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}^{n,m}\eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right)\right] \mathrm{d}v\mathrm{d}u \right|.$$

Summing over $i, j$, then using the Cauchy–Minkowski inequality, we get

$$\left[\sum_{i,j} |(G_{ij})_{p,q}(\bar{A}) - G_{ij}(\bar{A})|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i,j} \int_0^1 \int_0^1 |\zeta_i(u)\gamma_j(v)|^2 \mathrm{d}v\mathrm{d}u\right]^{\frac{1}{2}}$$
$$\left[\sum_{i,j} \int_0^1 \int_0^1 \left| \Psi_{p,q}\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}^{n,m}\eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right) - \Psi\left(u, v, \left(\frac{F(u,v)}{\mu} + \frac{\lambda}{\mu}\sum_{l,k}^{n,m}\eta_l(u)\beta_k(v)\bar{A}_{lk}\right)\right)\right|^2 \mathrm{d}v\mathrm{d}u\right]^{\frac{1}{2}}.$$

Applying conditions (6) and (7), we have

$$\|(\bar{G})_{p,q}(\bar{A}) - \bar{G}(\bar{A})\| \leq \left|\frac{\lambda}{\mu}\right| C^*\epsilon^* = \epsilon. \tag{19}$$

The previous inequality (19) shows that $(\bar{G})_{p,q}(\bar{A}) \to \bar{G}(\bar{A})$ uniformly for all $\bar{A} \in S^*_\beta$. To prove the compactness of $\bar{G}$, we let $\{A_{p,q}\}$ be any sequence in $S^*_\beta$. Then, we can choose a subsequence $\{A_{p_1,q}\}$ such that $\{(\bar{G})_{p_1,q}A_{p_1,q}\}$ converges. From that subsequence, we can extract a new subsequence $\{A_{p_1,q_1}\}$ in which $\{(\bar{G})_{p_1,q_1}A_{p_1,q_1}\}$ converges, and so on.

Thus, we obtain a chain of subsequences,

$$\{A_{p,q}\} \supset \{A_{p_1,q}\} \supset \{A_{p_1,q_1}\} \supset \cdots \supset \{A_{p_j,q_l}\} \supset \cdots,$$

such that the sequence $\{(\bar{G})_{p_i,q_k}A_{n_j,m_l}\}$ converges for all $i = 1, 2, \ldots, j$ and $k = 1, 2, \ldots, l$. Finally, we pick the sequence $\{A_{p_p,q_q}\}$, which is a subsequence of every $A_{p_i,q_k}$ except for a finite number of elements, and clearly $\{(\bar{G})_{p_i,q_k}A_{p_p,q_q}\}$ converges for every $i, k$.

Now,

$$\|\bar{G}A_{p_p,q_q} - \bar{G}A_{o_o,r_r}\| = \|\bar{G}A_{p_p,q_q} - (\bar{G})_{p_i,q_k}A_{p_p,q_q} + (\bar{G})_{p_i,q_k}A_{p_p,q_q} - (\bar{G})_{p_i,q_k}A_{o_o,r_r} + (\bar{G})_{p_i,q_k}A_{o_o,r_r} - \bar{G}\Phi_{o_o,r_r}\|.$$

Since $\|(\bar{G})_{p_i,q_k}A_{p_p,q_q} - (\bar{G})_{p_j,q_k}A_{o_o,r_r}\| \to 0$ as $p, q, o, r \to \infty$, for large $i, k$, one gets

$$\|\bar{G}A_{p_p,q_q} - \bar{G}A_{o_o,r_r}\| \leq 2\epsilon, \quad \forall p, o > p_0(\epsilon),\ q, r > q_0(\epsilon).$$

Hence, the sequence $\{\bar{G}A_{p,q}\}$ is a Cauchy sequence, so $\bar{G}(S^*_\beta)$ is compact.

**Table 1**
Example 1.

| $(x, y)$ | $\Phi$ | $\Phi_N^{(NL)}$ | $\Phi_N^{(L)}$ | $\lvert\Phi - \Phi_N^{(NL)}\rvert$ | $\lvert\Phi - \Phi_N^{(L)}\rvert$ |
|---|---|---|---|---|---|
| (0.1, 0.1) | 0.0001 | 0.0001 | 0.0001 | 1.0 E−15 | 2.1 E−15 |
| (0.1, 0.7) | 0.0049 | 0.0049 | 0.0049 | 1.0 E−15 | 2.7 E−15 |
| (0.3, 0.1) | 0.0009 | 0.0009 | 0.0009 | 9.0 E−15 | 1.89 E−14 |
| (0.3, 0.8) | 0.0576 | 0.0576 | 0.0576 | 9.0 E−15 | 2.52 E−14 |
| (0.6, 0.2) | 0.0144 | 0.0144 | 0.0144 | 3.6 E−14 | 7.92 E−14 |
| (0.6, 0.9) | 0.2916 | 0.2916 | 0.2916 | 3.6 E−14 | 1.044 E−13 |
| (0.9, 0.3) | 0.0729 | 0.0729 | 0.0729 | 8.1 E−14 | 1.863 E−13 |
| (0.9, 0.5) | 0.2025 | 0.2025 | 0.2025 | 8.1 E−14 | 2.025 E−13 |

According to the previous lemmas, we see that $\bar{G}$ is a continuous operator that maps a closed convex set $S_\beta^*$ in $l_2 \times l_2$ into itself and that $\bar{G}(S_\beta^*)$ is a compact set. So, by the Schauder fixed point theorem, $\bar{G}$ has at least one fixed point in $S_\beta^*$. □

## 5. Examples

**Example 1.** Consider the NIE

$$\Phi(x, y) - 0.01 \int_0^1 \int_0^1 x^2 u(y + v^2)\Phi^\alpha(u, v)\mathrm{d}v\mathrm{d}u = F(x, y), \quad `1234567, \tag{20}$$

with exact solution $\Phi(x, y) = x^2y^2$. This equation represents a two-dimensional NIE of the second kind if $\alpha \neq 1$, and $\alpha$ is real.

In Table 1, we select some points $(x, y)$ in the domain of the definition of Eq. (20), and we list the values of its exact solution, $\Phi(x, y)$, its approximate solution $\Phi_N^{(NL)}(x, y)$ when $\alpha = 2$ (the nonlinear case), and its approximate solution when $\alpha = 1$ (the linear case) at these selected points. The table also contains the error in each case.

**Example 2.** Consider the NIE

$$\Phi(x, y) - 0.01 \int_0^1 \int_0^1 x^3y^3 \sin(xu) \cos(yv)\Phi^\alpha(u, v)\mathrm{d}u\mathrm{d}v = F(x, y), \tag{21}$$

with exact solution $xy$. The kernel has the form

$$K(x, u; y, v) = x^3 \sin(xu)y^3 \cos(yv) = x^3y^3 \left(xu - \frac{x^3u^3}{3!} + \cdots\right)\left(1 - \frac{y^2v^2}{2!} + \cdots\right). \tag{22}$$

Now, we take two approximations for this kernel.

1- The first approximation is $K_1(x, u; y, v) = y^3x^4u$. Let $\Phi_{N1}^{(L)}(x, y)$ be the approximate solution of Eq. (21)in the linear case ($\alpha = 1$) while $\Phi_{N1}^{(NL)}(x, y)$ is the first approximate solution of the same equation in the nonlinear case ($\alpha = 2$).
2- Assume that the second approximation for the kernel is

$$K_2(x, u; y, v) = \left(y^3 - \frac{y^5v^2}{2!}\right)\left(x^4u - \frac{x^6u^3}{3!}\right).$$

Let $\Phi_{N2}^{(L)}(x, y)$ be the second approximate solution of Eq. (21)in the linear case ($\alpha = 1$) while $\Phi_{N2}^{(NL)}(x, y)$ is the second approximate solution of the same equation in the nonlinear case ($\alpha = 2$).

Table 2 compares the exact solution of Eq. (21) with $\Phi_{N1}^{(NL)}(x, y)$ and $\Phi_{N1}^{(L)}(x, y)$ for different values of $x$ and $y$. The estimate error in each case is also tabulated.

Table 3 compares the exact solution of Eq. (21) with $\Phi_{N2}^{(NL)}(x, y)$ and $\Phi_{N2}^{(L)}(x, y)$ for different values of $x$ and $y$. The estimate error in each case is also tabulated.

**Example 3.** Consider the NIE

$$\Phi(x, y) - 0.01 \int_0^1 \int_0^1 xy^2 \cos(yv)\mathrm{e}^{-\Phi^2(u,v)}\mathrm{d}u\mathrm{d}v = F(x, y). \tag{23}$$

(*Exact solution* $= \sqrt{xy}$),

$$K(x, u; y, v) = xu \cos(yv) = xy^2 \left(1 - \frac{y^2v^2}{2!} + \cdots \right. \tag{24}$$

**Table 2**
Example 2.

| $(x, y)$ | $\Phi$ | $\Phi_{N1}^{(NL)}$ | $\Phi_{N1}^{(L)}$ | $\lvert\Phi - \Phi_{N1}^{(NL)}\rvert$ | $\lvert\Phi - \Phi_{N1}^{(L)}\rvert$ |
|---|---|---|---|---|---|
| (0.2, 0.2) | 0.04 | 0.07873693373 | 0.04000000034 | 0.03873693373 | 3.4E−10 |
| (0.2, 0.8) | 0.16 | 0.1892505533 | 0.1600002182 | 0.0292505533 | 2.1E−07 |
| (0.3, 0.5) | 0.15 | 0.1830182007 | 0.1500001216 | 0.0330182007 | 1.2E−07 |
| (0.5, 0.7) | 0.35 | 0.3740230775 | 0.3500051103 | 0.0240230775 | 5.1E−06 |
| (0.6, 0.8) | 0.48 | 0.4985568487 | 0.4800206204 | 0.0185568487 | 2.1E−05 |
| (0.8, 0.7) | 0.56 | 0.5725268140 | 0.5600412823 | 0.0125268140 | 4.1E−05 |

**Table 3**
Example 2.

| $(x, y)$ | $\Phi$ | $\Phi_{N2}^{(NL)}$ | $\Phi_{N2}^{(L)}$ | $\lvert\Phi - \Phi_{N2}^{(NL)}\rvert$ | $\lvert\Phi - \Phi_{N2}^{(L)}\rvert$ |
|---|---|---|---|---|---|
| (0.3, 0.6) | 0.18 | 0.179999 | 0.18 | 3.9 E−09 | 5.0 E−10 |
| (0.4, 0.2) | 0.08 | 0.079999 | 0.8 | 1.0 E−10 | 6.3 E10 |
| (0.5, 0.9) | 0.45 | 0.449999 | 0.44999 | 4.463 E−07 | 5.278 E−07 |
| (0.6, 0.3) | 0.18 | 0.179999 | 0.18 | 2.7 E−09 | 8.0 E−09 |
| (0.7, 0.1) | 0.07 | 0.069999 | 0.07 | 2.9 E−10 | 4.2 E−10 |
| (0.8, 0.8) | 0.64 | 0.639999 | 0.63999 | 1.47 E−06 | 1.6 E−06 |

**Table 4**
Example 3.

| $(x, y)$ | $\Phi$ | $\lvert\Phi - \Phi_{N1}\rvert$ | $\lvert\Phi - \Phi_{N2}\rvert$ | $\lvert\Phi - \Phi_{N3}\rvert$ |
|---|---|---|---|---|
| (0.1, 0.8) | 0.948683298 | 0.249007619 E−4 | 0.00001529 | 1.439668036454674 E−8 |
| (0.2, 0.5) | 0.836660026 | 0.000007858225 | 0.000002354 | 6.433499308432 E−10 |
| (0.3, 0.3) | 0.774596669 | 0.000001594094 | 0.000001792 | 1.6890366328 E−12 |
| (0.4, 0.2) | 0.774596669 | 0.00004482499 | 0.00000960 | 1.11144443706 E−11 |
| (0.7, 0.7) | 1.18321595 | 0.00010327240493 | 0.000011214 | 3.45860355133781 E−8 |
| (0.9, 0.6) | 1.22474871 | 0.0007245009762 | 0.00013104 | 1.28549325151989 E−8 |

Now, we take two approximations for $K(x, u; y, v)$.

1- Assume that the first approximation is $K_1(x, u; y, v) = xy^2$. For this assumption, we obtain the first approximate solution of Eq. (23), which is denoted by $\Phi_{N1}(x, y)$.
2- The second approximation is $K_2(x, u; y, v) = xy^2(1-\frac{y^2v^2}{2!})$, and the second approximate solution is denoted by $\Phi_{N2}(x, y)$.
3- The third approximation is $K_3(x, u; y, v) = xy^2(1-\frac{y^2v^2}{2!}+\frac{y^4v^4}{4!})$, and the corresponding approximate solution is denoted by $\Phi_{N3}(x, y)$.

Table 4 compares the errors in these three approximations, $|\Phi-\Phi_{N1}(x, y)|$, $|\Phi-\Phi_{N2}(x, y)|$, and $|\Phi-\Phi_{N3}(x, y)|$, for different values of $x$ and $y$. We found that the error $|\Phi - \Phi_{N3}(x, y)|$ is much better than the others. Moreover,

$$\max_{(x,y)}\left\{\frac{|\Phi-\Phi_{N2}(x, y)|}{|\Phi-\Phi_{N3}(x, y)|}\right\} > \max_{(x,y)}\left\{\frac{|\Phi-\Phi_{N1}(x, y)|}{|\Phi-\Phi_{N2}(x, y)|}\right\} >> 3.$$

## 6. Conclusion

We deduce the following.

1- When the kernel of the integral equation is degenerate, the degenerate kernel method gives directly the exact solution for the integral equation of the second kind in the linear case and the nonlinear case, while the other methods fail; see Example 1 (the kernel in this example is $k(x, y, u, v) = x^2u(y + v^2)$).
2- The error decreases as $n$ and $m$ increase, where the maximum value of the error in the nonlinear case at $x = 1, y = 1$ for $n = m = 1$ is 0.0003, while, for $n = m = 2$, the maximum value of the error is $1.4 \times 10^{-5}$, and so on.
3- In Example 2, the maximum value of the error in the linear case at $x = 1, y = 1$ when $n = m = 2$ is $2.5 \times 10^{-5}$, which is more than the maximum value of the error in the corresponding nonlinear case, which is $1.4 \times 10^{-5}$.
4- In Example 3, we take the known nonlinear function $e^{-\phi^2}$, which is more complicated than the previous two examples, and we note that in this case the error is stable with increasing $n$, $m$.
5- For a continuous kernel, the degenerate kernel method is considered the best method to obtain the solution of linear and nonlinear integral equations of the second kind in one and two dimensions.
6- Storage requirements: writing the kernel in a degenerate form requires a storage $O(nm)$, while the algebraic system requires $O(n^2m^2)$.
7- The performance of our method mainly depends on the method used to solve the algebraic linear system (when $\alpha = 1$) or the nonlinear system (when $\alpha \neq 1$).

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